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Vanishing Spontaneous Magnetization for Quantum Mechanical Models of a Spin Glass

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We prove that the spontaneous magnetization vanishes identically (independently of boundary conditions) for certain quantum mechanical models of a spin glass.

KEY WORDS: Spin glass; spontaneous magnetization; Elitzur's theorem; Mattis-xy and Mattis–Heisenberg models.

Elitzur's theorem,(1) originally proved in the framework of lattice gauge theories, provides one of the most general results for spin glasses, namely, that the spontaneous magnetization vanishes identically, independently of boundary conditions. Stated thus, the theorem was proved for a large class of classical (i.e., Ising-like) models in Ref. 3 (see also Ref. 2). In this note we generalize the theorem to a class of quantum mechanical models of a spin glass. In addition to possible structural clarification, the subject may not be entirely academic because the quantum mechanical nature is essential for the Kondo effect (see, e.g., Ref. 4, Chap. 6.8) and it might also be of relevance in the spin glass regime (see also the discussion in Ref. 4, Chap. 6.9). A theory of classical spin glasses has been developed recently,(5) and order parameters have been analyzed in Ref. 6.

We consider for simplicity models described by the Hamiltonians

$$H_\Lambda(\{J\}) = -\tfrac{1}{2} \sum_{i,j\in\Lambda} J(i,j)\phi(i-j)g(\{S_i\},\{S_j\}) \tag{1}$$

where Λ is some finite region in $\mathbb{Z}^\nu$ (ν being the spatial dimension). Above,

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Φ is a (possibly long-range) potential and $J(i,j)=J(j,i)$, $J(i,i)=0$ are real random variables in some probability space, denoted collectively by $\{J\}$. In addition, $S_i^{(\alpha)}$, $\alpha=1,2,3$, are spin operators corresponding to a fixed value S of the total spin at each site, satisfying

$$\left[S_i^{(\alpha)}, S_j^{(\beta)}\right] = i \in^{\alpha\beta\gamma} S_i^{(\gamma)} \delta_{i,j}, \qquad i,j \in \mathbb{Z}^\nu$$

and $\{S_i\}$ denotes the set $\{S_i^{(\alpha)}, \alpha=1,2,3\}$. We require that ϕ, $\{J\}$ and the function of two variables g in (1) satisfy conditions sufficient for the existence of the thermodynamic limit.[7-9] Generalization to many-body interactions is straightforward.

In addition to the conditions stated in Ref. 7 or 9 we shall assume that the probability measure μ describing the distribution of the $\{J\}$ satisfies

$$\langle f(\{J\})\rangle_\mu = \langle f(\{-J\})\rangle_\mu \tag{2}$$

where $\{-J\}$ denotes a configuration obtained from $\{J\}$ by reversing the sign of an (arbitrary) number of coupling constants, and f is any μ-measurable function whose expectation $\langle f_\mu \rangle$ with respect to μ exists. We fix $\alpha_0 \in [1,3]$ and define the magnetization in the α_0 direction (a random variable) by

$$M_\Lambda(\{J\},h) = \frac{1}{|\Lambda|}\left\langle \sum_{i\in\Lambda} S_i^{(\alpha_0)} \right\rangle_{h,\{J\}}$$

where $|\Lambda|$ is the number of points in Λ and $\langle\cdot\rangle_{h,\{J\}}$ the Gibbs expectation value

$$\langle A\rangle_{h,\{J\}} \equiv \frac{1}{Z_\Lambda}\operatorname{tr}\{A \exp - \beta H_\Lambda(\{J\},h)\} \tag{3a}$$

$$Z_\Lambda \equiv \operatorname{tr}\exp[-\beta H_\Lambda(\{J\},h)] \tag{3b}$$

$$H_\Lambda(\{J\},h) \equiv H_\Lambda(\{J_\Lambda\}) - h\sum_{i\in\Lambda} S_i^{(\alpha_0)} \tag{3c}$$

the traces being taken over $\mathscr{H}_\Lambda \equiv \bigotimes_{i\in\Lambda}\mathbb{C}_i^{2S+1}$ and $h \geqslant 0$.

Proposition. Suppose that there exists for each site $i_0 \in \mathbb{Z}^\nu$ a unitary operator $U_{i_0}^{(\alpha_0)}$ on $\mathbb{C}_{i_0}^{2S+1}$ such that

$$U_{i_0}^{(\alpha_0)} S_{i_0}^{(\alpha_0)} U_0^{(\alpha_0)-1} = -S_{i_0}^{(\alpha_0)} \tag{4}$$

and

$$U_{i_0}^{(\alpha_0)} H_\Lambda(\{J\}) U_{i_0}^{(\alpha_0)-1} = H_\Lambda(\{-J_{i_0}\}) \qquad \forall i_0 \in \Lambda \tag{5a}$$

where $\{-J_{i_0}\}$ is the configuration defined by

$$-J_{i_0}(i,j) \equiv \begin{cases} J(i,j) & \text{if} \quad i \neq i_0 \quad \text{and} \quad j \neq i_0 \\ -J(i,j) & \text{if} \quad i = i_0 \quad \text{or} \quad j = i_0 \end{cases} \tag{5b}$$

Then

$$0 \leqslant \langle M_\Lambda(\{J\},h)\rangle_\mu \leqslant \beta h e^{2hS} S(S+1) \tag{6}$$

Remark. The main idea of the following simple argument is the following: Local gauge transformations $\{J\} \to \{-J_{i_0}\}$, $S_{i_0}^{(\alpha_0)} \to -S_{i_0}^{(\alpha_0)}$ cannot be spontaneously broken because the states in the ergodic decomposition[10] would differ only locally and hence correspond to unitarily equivalent representations of the spin algebra. See also Ref. 11.

Proof. By (2), (4), and (5)

$$\begin{aligned}\left\langle\left\langle S_{i_0}^{(\alpha_0)}\right\rangle_{h,\{J\}}\right\rangle_\mu &= \left\langle\frac{\operatorname{tr}\left[U_{i_0}^{(\alpha_0)}S_{i_0}^{(\alpha_0)}U_{i_0}^{(\alpha_0)-1}U_{i_0}^{(\alpha_0)}e^{-\beta H_\Lambda(\{J\},h)}U_{i_0}^{(\alpha_0)-1}\right]}{Z_\Lambda}\right\rangle_\mu \\ &= -\left\langle\left\langle S_{i_0}^{(\alpha_0)}\right\rangle_{h,\{J\}}\right\rangle_\mu \\ &\quad - \left\langle\frac{\operatorname{tr}\left\{S_{i_0}^{(\alpha_0)}\left[e^{-\beta[H_\Lambda(\{J\},h)+2hS_{i_0}^{(\alpha_0)}]} - e^{-\beta H_\Lambda(\{J\},h)}\right]\right\}}{Z_\Lambda}\right\rangle_\mu\end{aligned} \tag{7}$$

We use now the Duhamel formula (see, e.g., Ref. 12 and references given there)

$$\frac{d}{d\lambda}\,\frac{\operatorname{tr}\left[e^{-\beta(H+\lambda B)}A\right]}{Z} = -\beta(B,A)_\lambda\,\frac{Z_\lambda}{Z} \tag{8}$$

where $Z_\lambda \equiv \operatorname{tr} e^{-\beta(H+\lambda B)}$, $Z = Z_{\lambda=0}$ and

$$(B,A)_\lambda \equiv \frac{\int_0^1 dx\,\operatorname{tr}\left[e^{-x\beta(H+\lambda B)}Be^{-(1-x)\beta(H+\lambda B)}A\right]}{Z_\lambda}$$

is the Duhamel two-point function. Inserting (8) into (7) with $H = H_\Lambda(\{J\}, h)$, $B = 2hS_{i_0}^{(\alpha_0)}$ and $A = S_{i_0}^{(\alpha_0)}$, $(A,B) = (A,B)_{\lambda,\{J\}}$, we obtain

$$\left\langle\left\langle S_{i_0}^{(\alpha_0)}\right\rangle_{h,\{J\}}\right\rangle_\mu = \beta h\left\langle\int_0^1 d\lambda\left(S_{i_0}^{(\alpha_0)}, S_{i_0}^{(\alpha_0)}\right)_{\lambda,\{J\}}\frac{Z_\lambda}{Z}\right\rangle_\mu \tag{9}$$

where $\langle\cdot\rangle_{\lambda,\{J\}}$ is the expectation value in the Gibbs state defined by the Hamiltonian $H_\Lambda(\{J\},h) + 2\lambda h S_{i_0}^{(\alpha_0)}$. Note that (9) shows in particular that the left-hand side is nonnegative. By Bogoliubov's inequality (see, e.g., Ref. 12)

$$0 \leqslant \left(S_{i_0}^{(\alpha_0)}, S_{i_0}^{(\alpha_0)}\right)_{\lambda,\{J\}} \leqslant \left\langle S_{i_0}^{(\alpha_0)2}\right\rangle_{\lambda,\{J\}} \leqslant S(S+1) \tag{10}$$

By the Golden–Thompson inequality (see, e.g., Ref. 13, Theorem 4)

$$\frac{Z_\lambda}{Z} \leqslant \langle e^{2\lambda h S_{i_0}^{(\alpha_0)}} \rangle_{h,\{J\}} \leqslant \| e^{2hS_{i_0}^{(\alpha_0)}} \| = e^{2hS} \tag{11}$$

The above estimates being independent of $i_0 \in \Lambda$, (9), (10), and (11) yield the final assertion. ■

In the above proof we had to average the various quantities over the probability distribution. The *reproducibility* of the outcomes of most experiments on random systems requires that statements be true "irrespective of the sample," i.e., with probability one (see Ref. 14 for a careful discussion):

Theorem.

$$\lim_{h \to 0+} \lim_{|\Lambda| + \infty} M_\Lambda(\{J\}, h) \equiv 0$$

with probability one, under the assumption of the proposition.

Remark. We assume for simplicity the $|\Lambda|$ are a sequence of cubes of side L; $|\Lambda| \to \infty$ means $L \to \infty$.

Proof. By Refs. 7 and 9, the sequence of free energies

$$f_\Lambda(\{J\}, h) \frac{+\beta^{-1}}{|\Lambda|} \log \operatorname{tr} e^{-\beta H_\Lambda(\{J\}, h)} \tag{12}$$

(equal to *minus* the usual free energies, for convenience) converges as $|\Lambda| \to \infty$ with probability one. The limiting free energy $f(\{J\}, h)$ is almost everywhere equal to $\langle f(J\}, h) \rangle_\mu$.[7,9]

Hence

$$f_\Lambda(\{J\}, h) \xrightarrow[|\Lambda| \to \infty]{} \langle f(\{J\}, h) \rangle_\mu \qquad \text{with probability one} \tag{13}$$

and

$$\langle f_\Lambda(\{J\}, h) \rangle_\mu \xrightarrow[|\Lambda| \to \infty]{} \langle f(\{J\}, h) \rangle_\mu \tag{14}$$

By the proposition, if $\epsilon \geqslant 0$,

$$0 \leqslant \langle f_\Lambda(\{J\}, h + \epsilon) \rangle_\mu - \langle f_\Lambda(\{J\}, h) \rangle_\mu \leqslant c\epsilon(h + \epsilon) \exp\left[2S(h + \epsilon)\right] \tag{15}$$

for some constant c independent of $h\Lambda$. Taking the limit $|\Lambda| \to \infty$ in (15) and using (14), we obtain

$$0 \leqslant \frac{d^+ \langle f(\{J\}, h) \rangle}{dh} \leqslant ch \exp(2Sh) \tag{16}$$

where $d^+ g(x)/dx$ denotes the right derivative of the function g at the point x: note that $\langle f(\{J\}, h) \rangle$ has everywhere a right derivative because it is a convex function of h. Also $f_\Lambda(\{J\}, h)$ are a sequence of convex functions of h and by (13), positivity of $M_\Lambda(\{J\},h)$ and Griffith's Lemma [15]:

$$0 \leqslant \liminf_{|\Lambda|\to\infty} M_\Lambda(\{J\},h) \leqslant \limsup_{|\Lambda|\to\infty} M_\Lambda(\{J\},h) \leqslant \frac{d^+\langle f(\{J\},h)\rangle_\mu}{dh} \tag{17}$$

for any $h > 0$, with probability one. The final assertion follows from (16) and (17) upon taking the limit $h \to 0_+$ ■

The conditions of the proposition are met by the generalized xy spin glass

$$H_\Lambda(\{J\}) = \frac{1}{2} \sum_{i,j\in\Lambda} J(i,j)\phi(i-j)\left[S_i^{(1)}S_j^{(1)} + S_i^{(2)}S_j^{(2)}\right]$$

with $\alpha_0 = 1$ or 2, choosing $U_{i_0}^{(\alpha_0)} = \exp(\pm i\pi S_{i_0}^{(3)})$. They do not apply to the full Heisenberg spin glass, for the obvious reason that there exists no unitary which implements $S_{i_0}^{(\alpha)} \to -S_{i_0}^{(\alpha)}$ for all $\alpha = 1,2,3$. This is the same reason why the Mattis-xy, but not the Mattis Heisenberg model, is reducible to an ordered model by a local guage transformation (Ref. 4, Chap. 6.9), and suggests that Heisenberg spin glasses *might* have different thermodynamic properties.

If (2) does not hold, random ferromagnetism may of course occur.[3] Another way to obtain a nonzero spontaneous magnetization is through a random external magnetic field. This was shown recently for the spherical model if $\nu \geqslant 5$ and the variance of the field is sufficiently small.[16]

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